

Existence, Uniqueness and stability of Mild Solution of Lipschitzian Quantum Stochastic Differential Equations

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Abstract

We introduce the concept of a mild solution of Lipschitzian quantum stochastic differential equations (QSDEs). Results on existence, uniqueness and stability of mild solution of QSDEs are established. This is accomplished within the framework of the Hudson-Parthasarathy formulation of quantum stochastic calculus. Here, the results on a mild solution are weaker compared with the ones in the literature.

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1 Introduction

Recent literatures reveal consistent study of existence of mild solution of differential equations ranging from classical differential equations to non-classical differential equations. For details, see [4, 5, 11-12, 16] and the references therein.

One of the main analytical difficulties in the theory of both classical and non classical stochastic differential equations (SDEs) arises when the coefficients driving the equation consist of unbounded operators, see [7]. In [6],

Balasubramaniam et al. discussed the existence of mild solutions for similinear neutral functional evolution equations with nonlocal conditions by using fractional power of operators and Krasnoselskii fixed point theorem.

In [4], existence and uniqueness of the mild solutions for stochastic differential equations for Hilbert valued stochastic processes are discussed, with the multiplicative noise term given by an integral with respect to a general compensated Poisson random measure were established.

When considering quantum stochastic differential equations (QSDEs) within the framework of Hudson and Parthasarathy [15] formulation of QSDEs not much has been done in this area. However, some properties of solution sets of quantum stochastic differential inclusions were established in [1, 2, 8].

Existence of mild solution of impulsive QSDE and SDE were also considered in [16, 18] using the fixed point theorem method. In [9], results on solution of impulsive QSDEs and the associated Kurzweil equations were established. The recurrence of such problems in the literature is the motivation for this work. Hence the results here will be an extension of the results on QSDEs in the literature.

We organize the rest of the paper as follows: In section 2, we adopt some definitions and notations of Ekhaguere's formulations in [10] and [8-9]. In sections 3 we introduce the QSDE with an infinitesimal generator of a family of simigroup and establish the main results on existence and uniqueness of solution. In section 4, we establish result on stability of solution. The methods we used here are adoption of similar methods applied in [10].

All through the remaining sections, as in the references [8-10], we employ the locally convex topological space $\tilde{\mathcal{A}}$ of non commutative stochastic processes. We also adopt the definitions and notations of the following spaces $clos(\mathcal{N})$, $clos(\tilde{\mathcal{A}})$, $Ad(\tilde{\mathcal{A}})$, $Ad(\tilde{\mathcal{A}})_{wac}$, $L_{loc}^p(\tilde{\mathcal{A}})$, $L_{loc}^p(\tilde{\mathcal{A}})$, $L_{\gamma,loc}^\infty$, the integrator processes Λ_π, A_g^+, A_f , for $f, g \in L_{\gamma,loc}^\infty(\mathbb{R}_+)$, $\pi \in L_{B(\gamma),loc}^\infty(\mathbb{R}_+)$.

Let $E, F, G, H \in L_{loc}^2(\tilde{\mathcal{A}} \times I)$. The following equation is the Hudson - Parthasarathy quantum stochastic differential equation in integral form in-

troduced in [10]:

$$\begin{aligned}
X(t) &= X_0 + \int_{t_0}^t (E(X(s), s)d\wedge_{\pi}(s) + F(X(s), s)dA_f(s) \\
&\quad + G(X(s), s)dA_g^+(s) + H(X(s), s)ds), \\
X(t_0) &= X_0, \quad t \in I
\end{aligned} \tag{1.1}$$

In equation (1.1), the coefficients E, F, G, and H lie in a certain class of stochastic processes for which quantum stochastic integrals against the gauge, creation, annihilation processes $\Lambda_{\Pi}, A_{f+}, A_g$ and the Lebesgue measure t are defined in [10]. In the work of [10], the Hudson and Parthasarathy [15] formulation of quantum stochastic calculus was employed to establish the equivalent form of quantum stochastic differential equation (1.1) given by

$$\begin{aligned}
\frac{d}{dt} \langle \eta, X(t)\xi \rangle &= P(X(t), t)(\eta, \xi) \\
\langle \eta, X(0)\xi \rangle &= \langle \eta, x_0\xi \rangle \quad \text{for almost all } t \in I
\end{aligned} \tag{1.2}$$

where η, ξ lie in some dense subspaces of some Hilbert spaces which has been defined in [10]. For the explicit form of the map $P(x, t) \rightarrow P(x, t)(\eta, \xi)$ appearing in equation (1.2), see [10]. Equation (1.2) is a first order non-classical ordinary differential equation with a sesquilinear form valued map P as the right hand side. Equation (1.1) is known to have a unique weakly absolutely continuous adapted solution $\Phi : I \rightarrow \tilde{\mathcal{A}}$ for the Lipschitzian coefficients E, F, G, H.

2 Fundamental Concepts and Notations

In what follows, we employ the locally convex topological state space $\tilde{\mathcal{A}}$ of noncommutative stochastic processes and we also adopt the definitions and notations. See the references [8-10] and the references therein.

2.1 Notation. In what follows, \mathcal{D} is some inner product space with \mathcal{R} as its completion, and γ is some fixed Hilbert space.

- (i) For each $t \in \mathbb{R}_+$, we write $L_\gamma^2(\mathbb{R}_+)$ (Resp. $L_\gamma^2([0, t])$; resp. $L_\gamma^2([t, \infty))$), for the Hilbert space of square integrable, γ -valued maps on $\mathbb{R}_+ \equiv [0, \infty)$ (resp. $[0, t]$; resp. $[t, \infty)$).
- (ii) The noncommutative stochastic processes which we shall discuss are densely defined linear operators on $\mathcal{R} \otimes \Gamma(L_\gamma^2(\mathbb{R}_+))$; the inner product of this complex Hilbert space will be denoted by $\langle \cdot, \cdot \rangle$ and its norm by $\| \cdot \|$.

2.2 Definition. For $\eta, \xi \in \underline{\mathcal{D}} \otimes \underline{\mathcal{E}}$, we define $\| \cdot \|_{\eta\xi}$ on \mathcal{A} by

$$\|x\|_{\eta\xi} = |\langle \eta, x\xi \rangle|, x \in \mathcal{A}.$$

Then $\{\| \cdot \|_{\eta\xi}, \eta, \xi \in \underline{\mathcal{D}} \otimes \underline{\mathcal{E}}\}$ is a family of seminorms on \mathcal{A} ; we write τ_w for the locally convex Hausdorff topology on \mathcal{A} determined by this family.

2.3 Notation. We denote by $\tilde{\mathcal{A}}$ the completions of the locally convex spaces (\mathcal{A}, τ_w) .

2.4 Notation. (i) We denote the space of sesquilinear forms on $\underline{\mathcal{D}} \otimes \underline{\mathcal{E}}$ by $\text{sesq}(\underline{\mathcal{D}} \otimes \underline{\mathcal{E}})$.

(ii) Let $I \subseteq \mathbb{R}_+$, we denote by $L^0(I, \underline{\mathcal{D}} \otimes \underline{\mathcal{E}})$ the set of all $\text{sesq}(\underline{\mathcal{D}} \otimes \underline{\mathcal{E}})$ -valued maps on I . i.e., $L^0(I, \underline{\mathcal{D}} \otimes \underline{\mathcal{E}}) = \{u : I \rightarrow \text{sesq}(\underline{\mathcal{D}} \otimes \underline{\mathcal{E}})\}$.

2.5 Definition. A member $z \in L^0(I, \underline{\mathcal{D}} \otimes \underline{\mathcal{E}})$ is:

- (i) absolutely continuous if the map $t \rightarrow z(t)(\eta, \xi)$ is absolutely continuous for arbitrary $\eta, \xi \in \underline{\mathcal{D}} \otimes \underline{\mathcal{E}}$.
- (ii) of bounded variation if over all partition $\{t_j\}_{j=0}^n$ of I ,

$$\sup_{\mathcal{H}} \left(\sum_{j=1}^n |z(t_j)(\eta, \xi) - z(t_{j-1})(\eta, \xi)| \right) < \infty$$

2.6 Definition. A stochastic process Φ will be called locally absolutely p -integrable if the map $t \rightarrow \|\Phi(t)\|_{\eta\xi}, t \in \mathbb{R}_+$, lies in $L_{loc}^p(I)$ for arbitrary $\eta, \xi \in \underline{\mathcal{D}} \otimes \underline{\mathcal{E}}$ and $p \in (0, \infty)$.

2.7 Notation. For $p \in (0, \infty)$ and $I \subseteq \mathbb{R}_+$, $L_{loc}^2(I \times \tilde{\mathcal{A}})$ denotes the set of maps $\Phi : I \times \tilde{\mathcal{A}} \rightarrow \tilde{\mathcal{A}}$ such that the map $t \rightarrow \Phi(X(t), t)$ lies in $L_{loc}^p(\tilde{\mathcal{A}})$ for every $X \in L_{loc}^p(\tilde{\mathcal{A}})$.

2.8 Definition: Let $I \subseteq \mathbb{R}_+$

(i) A map $\Phi : I \times \tilde{\mathcal{A}} \rightarrow \tilde{\mathcal{A}}$ will be called Lipschitzian if for any $\eta, \xi \in \mathcal{D} \otimes \mathcal{E}$, there exists a function

$$K_{\eta\xi}^\Phi : I \rightarrow (0, \infty)$$

lying in $L_{loc}^1(I)$ such that,

$$\|\Phi(x, t) - \Phi(y, t)\|_{\eta\xi} \leq K_{\eta\xi}^\Phi(t) \|x - y\|_{\eta\xi}$$

for all $x, y \in \tilde{\mathcal{A}}$ and almost all $t \in I$.

(ii) If for $\eta, \xi \in \mathcal{D} \otimes \mathcal{E}$, $\Phi_{\eta\xi}$ is a map from $I \times \tilde{\mathcal{A}}$ into $\text{ses}[\mathcal{D} \otimes \mathcal{E}]$ then for $(x, t) \in I \times \tilde{\mathcal{A}}$, the value of $\Phi(x, t)$ at $\eta, \xi \in \mathcal{D} \otimes \mathcal{E}$ will be denoted by $\Phi(x, t)(\eta, \xi)$. Such a map will be called Lipschitzian if for arbitrary $\eta, \xi \in \mathcal{D} \otimes \mathcal{E}$

$$|\Phi(x, t)(\eta, \xi) - \Phi(y, t)(\eta, \xi)| \leq K_{\eta\xi}^\Phi(t) \|x - y\|_{\eta\xi}$$

for all $x, y \in \tilde{\mathcal{A}}$ and almost all $t \in I$.

(iii) If Φ is a map from $I \times \tilde{\mathcal{A}}$ into the $\text{sesq}(\mathcal{D} \otimes \mathcal{E})$ then for $(x, t) \in I \times \tilde{\mathcal{A}}$, the value of $\Phi(x, t)$ at $\eta, \xi \in \mathcal{D} \otimes \mathcal{E}$, will be called Lipschitzian (resp. continuous) if for arbitrary $\eta, \xi \in \mathcal{D} \otimes \mathcal{E}$, the map $(x, t) \rightarrow \Phi(x, t)(\eta, \xi)$ from $I \times \tilde{\mathcal{A}}$ to \mathbb{C} is Lipschitzian (resp. continuous).

2.9 Definition. Let $I \subseteq \mathbb{R}_+$.

(i) By a stochastic process indexed by I , we mean a function on I with values in $\text{clos}(\tilde{\mathcal{A}})$.

(ii) And for $p \in (0, \infty)$, $L_{loc}^p(I \times \tilde{\mathcal{A}})$ is the set of maps $\Phi : I \times \tilde{\mathcal{A}} \rightarrow \text{clos}(\tilde{\mathcal{A}})$ such that $t \rightarrow \Phi(X(t), t)$, $t \in I$ lies in $L_{loc}^p(\tilde{\mathcal{A}})_{mvs}$ for every $X \in L_{loc}^p(\tilde{\mathcal{A}})$.

2.10 Definition. A stochastic process $p \in Ad(\tilde{\mathcal{A}})$ is called simple if there exists an increasing sequence $t_n, n = 0, 1, 2, \dots$ with $t_0 = 0$ and $t_n \rightarrow \infty$ such that for each $n \geq 0$,

$$p(t) = p(t_n) \quad \text{and} \quad t \in [t_n, t_{n+1})$$

For a topological space \mathcal{N} , let $clos(\mathcal{N})$ be the collection of all nonempty closed subsets of \mathcal{N} ; we shall employ the Hausdorff topology on the $clos(\tilde{\mathcal{A}})$ as defined in [10]. Also, for $A, B \in clos(\mathbb{C})$ and $x \in \mathbb{C}$, we define the Hausdorff distance, $\rho(A, B)$ as in [18]. Then ρ is a metric on the $clos(\mathbb{C})$ and induces a metric topology on the space.

2.11 Definition: A map $P : \tilde{\mathcal{A}} \times [t_0, T] \rightarrow sesq[\underline{\mathcal{D}} \otimes \underline{\mathcal{E}}]$ belongs to the class $C(\tilde{\mathcal{A}} \times [t_0, T], W)$ if for arbitrary $\eta, \xi \in \underline{\mathcal{D}} \otimes \underline{\mathcal{E}}$

- (i) $P(x, \cdot)(\eta, \xi)$ is measurable for each $x \in \tilde{\mathcal{A}}$
- (ii) There exists a family of measurable functions $M_{\eta\xi} : [t_0, T] \rightarrow \mathbb{R}_+$ such that $\int_{t_0}^t M_{\eta\xi} ds < \infty$ and $|P(x, \cdot)(\eta, \xi)| \leq M_{\eta\xi}(s)$, $(x, s) \in \tilde{\mathcal{A}} \times [t_0, T]$
- (iii) There exists measurable functions $K_{\eta\xi} : [t_0, T] \rightarrow \mathbb{R}_+$ such that for each $t \in [t_0, T]$, $\int_{t_0}^t K_{\eta\xi} ds < \infty$, and

$$|P(x, s)(\eta, \xi) - P(y, s)(\eta, \xi)| \leq K_{\eta\xi}^p(s)W(\|x - y\|_{\eta\xi})$$

For $(x, s), (y, s) \in \tilde{\mathcal{A}} \times [t_0, T]$ and where for (i) -(iii) $W(t) = t$. and

$$h_{\eta\xi}(t) = \int_{t_0}^t M_{\eta\xi}(s)ds + \int_{t_0}^t K_{\eta\xi}(s)ds$$

2.12 Notation:

The class $C(\tilde{\mathcal{A}} \times [t_0, T], W)$, denotes the class of sesquilinear form-valued maps which satisfy the Lipschitz condition and the Caratheodory conditions.

2.13 Definition. A member $z \in L^0(I, \underline{\mathcal{D}} \otimes \underline{\mathcal{E}})$ is:

- (i) absolutely continuous if the map $t \rightarrow z(t)(\eta, \xi)$ is absolutely continuous for arbitrary $\eta, \xi \in \mathcal{D} \otimes \mathcal{E}$.
- (ii) of bounded variation if over all partition $\{t_j\}_{j=0}^n$ of I ,

$$\sup_{\mathcal{H}} \left(\sum_{j=1}^n |z(t_j)(\eta, \xi) - z(t_{j-1})(\eta, \xi)| \right) < \infty$$

- (iii) of essentially bounded variation if z is equal almost everywhere to some member of $L^0(I, \mathcal{D} \otimes \mathcal{E})$ of bounded variation.
- (iv) A stochastic process $x \in L^0(I, \tilde{\mathcal{A}})$ is of bounded variation if

$$\sup_{\mathcal{H}} \left(\sum_{j=1}^n | \langle \eta, x(t_j)\xi \rangle - \langle \eta, x(t_{j-1})\xi \rangle | \right) < \infty.$$

for arbitrary $\eta, \xi \in \mathcal{D} \otimes \mathcal{E}$ and where supremum is taken over all partitions $\{t_j\}_{j=0}^n$ of I .

2.14 Notation. We denote by $BV(\tilde{\mathcal{A}})$ the set of all stochastic processes of bounded variation on I .

2.15 Definition. For $x \in BV(\tilde{\mathcal{A}})$, define for arbitrary $\eta, \xi \in \mathcal{D} \otimes \mathcal{E}$,

$$Var_{[a,b]} x_{\eta\xi} = \sup_{\tau} \left(\sum_{j=1}^n \|x(t_j) - x(t_{j-1})\|_{\eta\xi} \right)$$

where τ is the collection of all partitions of the interval $[a, b] \subset I$. If $[a, b] = I$, we write $Var_I x_{\eta\xi} = Var x_{\eta\xi}$. Then $\{Var x_{\eta\xi} : \eta, \xi \in \mathcal{D} \otimes \mathcal{E}\}$ is a family of seminorms which generates a locally convex topology on $BV(\tilde{\mathcal{A}})$.

2.16 Notation.

- (i) We denote by $\overline{BV}(\tilde{\mathcal{A}})$ the completion of $BV(\tilde{\mathcal{A}})$ in the said topology.
- (ii) We denote by $A := BV(\tilde{\mathcal{A}}) \cap Ad(\tilde{\mathcal{A}})_{wac}$ the stochastic process that are weakly, absolutely continuous and of bounded variation on $[t_0, T]$. The space $A(\eta, \xi)$ equipped with the norm

$$\|x\|_{PC} = \sup\{|x(t)(\eta, \xi)| : t \in I\}$$

is a Banach space.

It has been well established that the quantum stochastic differential equation (2.1) introduced by Hudson and Parthasarathy provide an essential tool in the theoretical description of physical systems, especially those arising in quantum optics, quantum measure theory, quantum open systems and quantum dynamical systems. See the [8-9] and the references therein.

3 Existence of Solution

Let A be the infinitesimal generator of a family of semigroup $\{T(t) : t \geq 0\}$ defined in [18]. We consider the existence of a mild solution of the quantum stochastic evolution problem given by

$$\begin{aligned} dx(t) &= A(t)x(t) + (E(x(t), t)d \wedge_{\pi}(t) + F(x(t), t)dA_g(t) \\ &\quad + G(x(t), t)dA_{f^+}(t) + H(x(t), t)dt), \\ &\text{almost all } t \in I = [0, T] \end{aligned} \quad (3.1)$$

$$x(0) = x_0.$$

The equivalent form of (3.1) is then given by

$$\begin{aligned} \frac{d}{dt} \langle \eta, X(t)\xi \rangle &= A(t)x(t) + P(x(t), t)(\eta, \xi) \\ x(t_0) &= x_0, \quad t \in [t_0, T] \end{aligned} \quad (3.2)$$

3.1 Definition: An adapted stochastic process $x \in A$ is called a mild solution of equation (3.1) if

$$\begin{aligned} \langle \eta, x(t)\xi \rangle - \langle \eta, T(t)x_0\xi \rangle &= \int_{t_0}^t T(t-s)(P(x(s), s)ds)(\eta, \xi) \\ x(t_0) &= x_0, \quad t \in [t_0, T] \end{aligned} \quad (3.3)$$

holds for every $s, t \in [t_0, T]$ identically. The map P in equation (3.2) is a sesquilinear form valued map defined in [10]. Equation (3.1) is understood in integral form (3.3) via its solution.

3.2 Definition: Let X be a banach space. A one parameter family $T(t)$, $0 \leq t < \infty$, of bounded linear operators from X into X is a semigroup of bounded linear operator on X if

- (i) $T(0) = I$, (I is the identity operator on X).
- (ii) $T(t + s) = T(t)T(s)$ for every $t, s \geq 0$ (the semigroup property).

The following theorem established in [10] will be useful in establishing the major result in this section.

3.1 Theorem. Let p, q, u, v be simple adapted stochastic processes in $Ad(\tilde{\mathcal{A}})$ and let M be their stochastic integral. If $\eta, \xi \in \mathcal{D} \otimes \mathcal{E}$ with $\eta = c \otimes e(\alpha), \xi = d \otimes e(\beta), c, d \in \mathcal{D}, \alpha, \beta \in L_{\gamma, loc}^{\infty}(\mathbb{R}_+)$, and $t \geq 0$, then

$$\begin{aligned} \langle \eta, M(t)\xi \rangle &= \int_0^t \langle \eta, \{ \langle \alpha(s), \pi(s)\beta(s) \rangle_{\gamma} p(s) \\ &\quad + \langle f(s), \beta(s) \rangle_{\gamma} q(s) + \langle \alpha(s), g(s) \rangle_{\gamma} u(s) + v(s) \} \xi \rangle ds \end{aligned} \quad (3.4)$$

Next, we establish a major result.

3.2 Theorem Assume that:

- (i) the coefficients E, F, G, H appearing in equation (3.2) satisfy the Lipschitz condition and belong to $L_{loc}^1(I \times \tilde{\mathcal{A}})$.
- (ii) there exists a constant N such that $\|T(t)\|_{\eta\xi} \leq N$ for each $t \geq 0$. Then for any fixed point $(X_0, t_0) \in \tilde{\mathcal{A}} \times I$ there exists a unique adapted and weakly absolutely continuous mild solution Φ of the quantum stochastic differential equation (3.1) satisfying $\Phi(t_0) = X_0$.

Proof. We first construct a τ_w - Cauchy sequence $\{\Phi_n(t)\}_{n \geq 0}$ of successive approximations of Φ in $\tilde{\mathcal{A}}$. All through except otherwise stated

$\eta, \xi \in \mathbb{D} \otimes \mathbb{E}$ is arbitrary. Fix $T > t_0$, $t \in [t_0, T]$. Define $T(t)\Phi_0(t) = NX_0$, and for $n \geq 0$

$$\begin{aligned} \Phi_{n+1}(t) &= NX_0 + \int_{t_0}^t T(t-s)(E(\Phi_n(s), s)d \wedge_{\pi}(s) \\ &\quad + F(\Phi_n(s), s)dA_g^+(s) + G(\Phi_n(s), s)dA_f(s) + H(\Phi_n(s), s)ds) \end{aligned}$$

We let each $\Phi_n(t)$, $n \geq 1$ define an adapted weakly absolutely continuous process in A .

By hypothesis, $E(X_0, s)$, $F(X_0, s)$, $G(X_0, s)$, and $H(X_0, s)$ belong to $\tilde{\mathcal{A}}_s$ for $s \in [t_0, T]$ and $E(X_0, \cdot)$, $F(X_0, \cdot)$, $G(X_0, \cdot)$, and $H(X_0, \cdot)$ lie in A . Therefore the quantum stochastic integral which defines $\Phi_1(t)$ exists for $t \in [t_0, T]$.

By equation (3.4), $\Phi_1(t)$ is weakly absolutely continuous and hence locally square integrable.

Assume now that $\Phi_n(t)$ is adapted and weakly absolutely continuous, then each $E(\Phi_n(s), s)$, $F(\Phi_n(s), s)$, $G(\Phi_n(s), s)$ and $H(\Phi_n(s), s)$ is adapted and lie in $L_{loc}^2(\tilde{\mathcal{A}})$. Thus $\Phi_{n+1}(t)$ is adapted and well defined. Again by equation(3.4), $\Phi_{n+1}(t)$ is a weakly absolutely continuous process in $L_{loc}^2(\tilde{\mathcal{A}})$. Hence we have proved our claim by induction. We consider the convergence of the successive approximations.

By equation (3.4) and the definition of the map P above, we have,

$$\begin{aligned} \|\Phi_{n+1}(t) - \Phi_n(t)\|_{\eta\xi} &= |\langle \eta, (\Phi_{n+1}(t) - \Phi_n(t))\xi \rangle| \\ &= N \left| \int_{t_0}^t (P(\Phi_n(s), s)(\eta, \xi) - P(\Phi_{n-1}(s), s)(\eta, \xi))ds \right| \end{aligned} \quad (1)$$

Since the coefficients E, F, G, H are Lipschitzian, then the map $(x, t) \rightarrow P(X, t)(\eta, \xi)$ is also Lipschitzian and hence satisfies

$$\begin{aligned} N|P(x, t)(\eta, \xi) - P(y, t)(\eta, \xi)| &\leq NK_{\eta\xi}^p(t)(\|x - y\|_{\eta\xi}) \\ &\forall x, y \in A, \quad t \in [t_0, T]. \end{aligned}$$

Substituting the last inequality in (1), we get

$$\|\Phi_{n+1}(t) - \Phi_n(t)\|_{\eta\xi} \leq \int_{t_0}^t NK_{\eta\xi}^p(s) (\|\Phi_n(s) - \Phi_{n-1}(s)\|_{\eta\xi}) ds \quad (2)$$

Since the map $s \rightarrow \|\Phi_1(s) - X_0\|_{\eta\xi}$ is continuous on $[t_0, T]$, we put

$$R_{\eta\xi} = \sup_{s \in [t_0, T]} \|\Phi_1(s) - X_0\|_{\eta\xi}, \quad s \in [t_0, T]$$

this implies that $\|\Phi_1(s) - X_0\|_{\eta\xi} \leq R_{\eta\xi}$. Also let

$$M_{\eta\xi}(t) = \int_{t_0}^t K_{\eta\xi}^p(s) ds$$

From (2) we have

$$\|\Phi_{n+1}(t) - \Phi_n(t)\|_{\eta\xi} \leq \frac{N(R_{\eta\xi})(M_{\eta\xi}(t))^n}{n!}, \quad n, i = 1, 2, \dots \quad (3)$$

This we prove by induction as follows.

For $n = 1$, inequality (3) holds by considering (2). Assume that (3) holds for $n = k$

i.e.

$$\|\Phi_{k+1}(t) - \Phi_k(t)\|_{\eta\xi} \leq \frac{N(R_{\eta\xi})(M_{\eta\xi}(t))^k}{k!}, \quad n = 1, 2, \dots \quad (4)$$

then by (2)

$$\begin{aligned} \|\Phi_{k+2}(t) - \Phi_{k+1}(t)\|_{\eta\xi} &\leq \int_{t_0}^t NK_{\eta\xi}^p(s) (\|\Phi_{k+1}(s) - \Phi_k(s)\|_{\eta\xi}) ds \\ &\leq \frac{N(R_{\eta\xi})}{k!} \int_{t_0}^t K_{\eta\xi}^p(s) (M_{\eta\xi}(s))^k ds \quad \text{by (4)} \end{aligned}$$

By applying integration by parts on the first term, we obtain

$$\int_{t_0}^t K_{\eta\xi}^p(s) (M_{\eta\xi}(s))^k ds = \frac{(M_{\eta\xi}(t))^{k+1}}{k+1} \quad (5)$$

Therefore,

$$\|\Phi_{k+2}(t) - \Phi_{k+1}(t)\|_{\eta\xi} \leq \frac{N(R_{\eta\xi})(M_{\eta\xi}(t))^{k+1}}{(k+1)!}$$

So that (3) holds for $n = k + 1$ and so holds for $n = 1, 2, 3, \dots$

Therefore, for any $n > k$,

$$\begin{aligned} \|\Phi_{n+1}(t) - \Phi_{k+1}(t)\|_{\eta\xi} &= N \|\sum_{m=k+1}^n (\Phi_{m+1}(t) - \Phi_m(t))\|_{\eta\xi} \\ &\leq N \sum_{m=k+1}^n \|\Phi_{m+1}(t) - \Phi_m(t)\|_{\eta\xi} \\ &\leq N \left[\sum_{m=k+1}^n \frac{(R_{\eta\xi})(M_{\eta\xi}(T))^m}{m!} \right] \end{aligned}$$

It follows that $\Phi_n(t)$ is a Cauchy sequence in $\tilde{\mathcal{A}}$ and converges uniformly to some $\Phi(t)$. Since $\Phi_n(t)$ is adapted and weakly absolutely continuous, the same is true of $\Phi(t)$.

We now show that $\Phi(t)$ satisfies the quantum stochastic differential equation (1.1).

Since $\Phi(t_0) = X(t_0) = X_0$, we have by equation (3.4),

$$\begin{aligned} &\left\| N \int_{t_0}^t [E(\Phi_n(s), s) d\wedge_\pi(s) + F(\Phi_n(s), s) dA_g^+(s) + G(\Phi_n(s), s) dA_f(s) + H(\Phi_n(s), s) ds] \right\|_{\eta\xi} \\ &- \left\| N \int_{t_0}^t [E(\Phi(s), s) d\wedge_\pi(s) + F(\Phi(s), s) dA_g^+(s) + G(\Phi(s), s) dA_f(s) + H(\Phi(s), s) ds] \right\|_{\eta\xi} \\ &= N \left| \int_{t_0}^t (P(\Phi_n(s), s)(\eta, \xi) - P(\Phi(s), s)(\eta, \xi)) ds \right| \\ &\leq N \int_{t_0}^t K_{\eta\xi}^p(s) (\|\Phi_n(s) - \Phi(s)\|_{\eta\xi}) \longrightarrow 0 \text{ as } n \longrightarrow \infty \end{aligned}$$

Since $\Phi_n(s) \longrightarrow \Phi(s)$ in $\tilde{\mathcal{A}}$ uniformly on $[t_0, T]$.

Thus

$$\begin{aligned} \Phi(t) &= \lim_{n \rightarrow \infty} \Phi_{n+1}(t) \\ &= T(t)X_0 + \lim_{n \rightarrow \infty} \left(\int_{t_0}^t T(t-s) (E(\Phi_n(s), s) d\wedge_\pi(s) + F(\Phi_n(s), s) dA_g^+(s) \right. \\ &\quad \left. + G(\Phi_n(s), s) dA_f(s) + H(\Phi_n(s), s) ds) \right) \\ &= T(t)X_0 + \int_{t_0}^t T(t-s) (E(\Phi(s), s) d\wedge_\pi(s) + F(\Phi(s), s) dA_g^+(s) \\ &\quad + G(\Phi(s), s) dA_f(s) + H(\Phi(s), s) ds). \end{aligned}$$

That is $\Phi(t), t \in [t_0, T]$ is a solution of equation (3.3).

Uniqueness of Solution

To establish the uniqueness of solution, we assume that $Y(t), t \in [t_0, T]$ is another adapted weakly absolutely continuous solution with $Y(t_0) = X_0$. Then, by equation (3.4), we obtain again

$$\begin{aligned} \|\Phi(t) - Y(t)\|_{\eta\xi} &= N \left| \int_{t_0}^t (P(\Phi(s), s)(\eta, \xi) - P(Y(s), s)(\eta, \xi)) ds \right| \\ &\leq \int_{t_0}^t N K_{\eta\xi}^p(s) (\|\Phi(s) - Y(s)\|_{\eta\xi}) ds. \end{aligned}$$

Since the integral $\int_{t_0}^t K_{\eta\xi}^p(s)$ exists on $[t_0, T]$, it is also essentially bounded on the given interval. Hence, there exists a constant $C_{\eta\xi, t}$ such that

$$\text{ess sup } K_{\eta\xi}^p(s) = C_{\eta\xi, t}, \quad s \in [t_0, T].$$

Thus

$$\|\Phi(t) - Y(t)\|_{\eta\xi} \leq N C_{\eta\xi, t} \int_{t_0}^t (\|\Phi(s) - Y(s)\|_{\eta\xi}) ds.$$

By the Gronwall's inequality, we conclude that $\Phi(t) = Y(t), t \in [t_0, T]$. Hence the solution is unique.

4 Stability of Solution

The next theorem establishes that the solutions equation (3.2) is stable. Hence we let the coefficients E, F, G, H satisfy the conditions of theorem 3.2. let $X(t), Y(t), t \in [t_0, T]$ be solutions to equation (3.2) corresponding to the initial conditions $T(t)X(t_0) = T(t)X_0$ and $T(t)Y(t_0) = T(t)Y_0$, respectively where $X_0, Y_0 \in A$. The solution $X(t)$ is stable under the changes in the initial condition $X(t_0) = X_0$ as follows:

Theorem 4.1: For given $T > t_0$ and $\epsilon > 0$, there exists $\delta > 0$ such that if $N \|X_0 - Y_0\|_{\eta\xi} < N\delta$, then $\|X(t) - Y(t)\|_{\eta\xi} < \epsilon$ still holds for all $t \in [t_0, T]$ and for each pair of $\eta, \xi \in \mathbb{D} \otimes \mathbb{E}$.

Proof: Let $X_n(t), Y_n(t)$, for $n=0, 1, \dots$ be the iterates corresponding to the initial conditions X_0 and Y_0 respectively, so that $X_0(t) = X_0$ and $Y_0(t) = Y_0$ for all $t_0 \leq t \leq T$. then we obtain the following estimate by employing the definition of P and equation (3.4) as in the proof of uniqueness of solution.

$$\begin{aligned} \|X_{n+1}(t) - Y_{n+1}(t)\|_{\eta\xi} &\leq \|T(t)X_0 - T(t)Y_0\|_{\eta\xi} \\ &\quad + \int_{t_0}^t T(t-s) (|P(X_n(s), s)(\eta, \xi) - P(Y_n(s), s)(\eta, \xi)|) ds \\ &= N \|X_0 - Y_0\|_{\eta\xi} + N(C_{\eta\xi, t} \int_{t_0}^t \|X_n(s) - Y_n(s)\|_{\eta\xi} ds), \end{aligned}$$

where $C_{\eta\xi, t}$ is the essential supremum of $K_{\eta\xi}^p(t)$ on $[t_0, T]$.

Therefore, by iteration, we obtain for $t_0 \leq t \leq T$

$$\begin{aligned} \|X_{n+1}(t) - Y_{n+1}(t)\|_{\eta\xi} &\leq N \|X_0 - Y_0\|_{\eta\xi} + N(C_{\eta\xi, t} \left[\int_{t_0}^t (\|X_0 - Y_0\|_{\eta\xi} \right. \\ &\quad \left. + C_{\eta\xi, t_1} \int_{t_0}^{t_1} \|X_{n-1}(t_2) - Y_{n-1}(t_2)\|_{\eta\xi} dt_2) dt_1 \right]) \\ &\leq N(\|X_0 - Y_0\|_{\eta\xi} + l_{\eta\xi}(t - t_0) \|X_0 - Y_0\|_{\eta\xi}) + \\ &\quad + N(l_{\eta\xi}^2 \int_{t_0}^t \int_{t_0}^{t_1} \|X_{n-1}(t_2) - Y_{n-1}(t_2)\|_{\eta\xi} dt_2 dt_1), \end{aligned}$$

where

$$l_{\eta\xi} = \max \{C_{\eta\xi, t}, C_{\eta\xi, t_1}\}.$$

Continuing the iteration and putting

$$L_{\eta\xi} = \max \{C_{\eta\xi, t}, C_{\eta\xi, t_j}, j = 1, 2, \dots, n\},$$

we obtain the estimate

$$\begin{aligned}
\| X_{n+1}(t) - Y_{n+1}(t) \|_{\eta\xi} &\leq N \|X_0 - Y_0\|_{\eta\xi} + NL_{\eta\xi}(t - t_0) \|X_0 - Y_0\|_{\eta\xi} + \dots \\
&+ NL_{\eta\xi}^n \frac{(t - t_0)^n}{n!} \|X_0 - Y_0\|_{\eta\xi} + \\
&+ NL_{\eta\xi}^{n+1} \int_{t_0}^t \int_{t_0}^{t_1} \dots \int_{t_0}^{t_n} \|X_0(t_{n+1}) - Y_0(t_{n+1})\|_{\eta\xi} dt_1 dt_2 \dots dt_{n+1} \\
&\leq N \sum_{m=0}^{n+1} \frac{L_{\eta\xi}^m}{m!} (t - t_0)^m \|X_0 - Y_0\|_{\eta\xi} \\
&\leq N \sum_{m=0}^{n+1} \frac{(L_{\eta\xi} T)^m}{m!} \|X_0 - Y_0\|_{\eta\xi} \\
&\leq N \|X_0 - Y_0\|_{\eta\xi} e^{(L_{\eta\xi} T)}.
\end{aligned}$$

Letting $n \rightarrow \infty$, we conclude that

$$\|X(t) - Y(t)\|_{\eta\xi} \leq N \|X_0 - Y_0\|_{\eta\xi} e^{(L_{\eta\xi} T)},$$

for all $t_0 \leq t \leq T$, and the result follows.

Conclusion

We have established existence, uniqueness and stability of mild solution of QSDE (3.1) via equation (3.2). This is possible since equivalence of these equations have been established in [10].

References

- [1] E.O. Ayoola, Exponential formula for the reachable sets of Quantum Stochastic Differential inclusions, Stoch. Anal. and Appl, 21(3)(2003), 515-543.
- [2] E. O. Ayoola, Topological properties of solution sets of Lipschitzian quantum stochastic differential inclusions, Acta Applicatae Mathematicae, 100(2008),15-37.
- [3] S. B. Agase and V. Raghavendra, Existence of mild solutions of similinear differential equations in Banach spaces, India J. Pure Appl. Maths. 21(9)(1990), 813-821.

- [4] S. Albeverio, V. Mandrekar and B. Rdiger, Existence of mild solutions for stochastic differential equations and semilinear equations with non-Gaussian Levy noise, *Stochastic Processes and their Applications* 119(3) (2009), 835-863.
- [5] G. Ballinger and X. Liu, Existence and uniqueness results for impulsive delay differential equations, *Dynam. contin. discrete impuls. systems* 5(1999), 579-591.
- [6] P. Balasubramanjam, J. Y. Park and A. V. Antony Kumar, Existence of solutions for similinear neutral Stochastic fractional differential equations with nonlocal conditions, *Nonlinear Anal.*, 71(2009), 1049-1058.
- [7] M. Benchohra and S. K. Ntouyas, Existence of mild solutions of second order initial value problems for differential inclusions with nonlocal conditions, *Atti Sem. Mat. Fis. Univ. Modena* 49(2)(2001), 351-361.
- [8] S. A. Bishop and T. A. Anake, Extension of Continuous Selection Sets to Non-Lipschitzian Quantum Stochastic Differential Inclusion, *J. Stochastic Analysis and Applications*, 31(2013), 1114-1124.
- [9] S. A. Bishop and O. O. Agboola, On Existence of solution for Impulsive Perturbed Quantum Stochastic Differential Equations and the associated Kurzweil equations, *IJRANSS*, 2(1)(2014), 51-60.
- [10] G. O. S. Ekhaguere, Lipschitzian Quantum Stochastic Differential Inclusions, *Int. journal of theoretical physics*, 31(11)(1992), 2003-2034.
- [11] Fang Li, Mild Solutions of Fractional Differential Equations with Nonlocal Conditions, *Advances in Difference Equations* 2010, **2010**:287861.

- [12] L. Fang and J. Zhang, Existence of Mild Solutions to Fractional Integrodifferential Equations Neutral Type with Infinite Delays, *Advances in Difference Equations* 2011, **2011**:287861.
- [13 T. Guendouzi and K. Mehdi, Existence of mild solutions for impulsive fractional stochastic equations with infinite delay, *Malaya journal of matematik* 4(1)(2013), 30-43.
- [14 H. L. Tidke and M. B. Dhakne, Existence And Uniqueness Of Mild Solutions Of Second Order Volterra Integrodifferential Equations With Nonlocal Conditions, *Applied Mathematics ENotes*, 9(2009), 101-108.
- [15] R. L. Hudson and K. R. Parthasarathy, Quantum Ito's formulae and stochastic evolutions, *Comm. math. phys.*, 93(1984), 301-324.
- [16 L. pan, Existence of Mild Solution for impulsive Stochastic Differential Equations with Nonlocal Conditions, *Differential Equations and Applications*. 4(3)(2010), 488-494.
- [17] A. pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.
- [18] M. O. Ogundiran and V. F. Payne, On the Existence and Uniqueness of solution of Impulsive Quantum Stochastic Differential Equation, *Differential equations and control processes*, N 2 (2013), *Electronic Journal*.