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Using the Average of the Extreme Values of a Triangular Distribution for a Transformation, and Its Approximant via the Continuous Uniform Distribution

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Abstract

This paper introduces a new probability distribution referred to as a transformed triangular distribution (TTD) by using the average of the extreme values (minimum and maximum) of the triangular distribution. The TTD is being approximated by the continuous uniform distribution. The basic moments of the TTD and those of the continuous uniform distribution are compared respectively, and a relationship established. This can be used in modeling and simulation.

Keywords: Moments, uniform distribution, triangular distribution, transformed distribution, continuous random variable.

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1 Introduction

Triangular distribution is a continuous distribution with a fixed minimum, fixed maximum and most likely value to occur (mode). The most likely value lies between the minimum and maximum values, forming a triangular-shaped distribution which shows that values near the minimum and maximum are less likely to occur [1]. The minimum and the maximum values are called the extreme values. The values can either be symmetrical (the $\text{mod}\, e = mean = median$) or asymmetrical [2].

The distribution is widely used to approximate Beta distribution [3]. Some of the earliest known written work on the triangular distribution are traceable to the work of [4,5], even though it was Simpson that first mentioned the distribution in his papers [6].

Nevertheless, many authors and researchers have worked on the triangular distribution; some of them are either on the statistical, probabilistic nature of the distribution or in the area of simulation and modeling. Some of the contributions include the characteristics of the distribution [7,8], the product of two identically independent distributed triangular variables [9], on the extension of the triangular distribution and the convolution [10,11], applications in project evaluation and review technique PERT [12-16], non-smooth sailing with respect to asymptotic distributions [17,18], Monte Carlo Simulation [19-22], the properties of bivariate triangular distribution [23-24], on the negative binomial-triangular distribution [25], advanced simulation and risk modeling [26-28] combination of triangular and exponential distributions [29], the sum of two triangular distributions [30], discrete nature of triangular distribution and non-parametric estimation for probability mass function [31].

The remaining part of the paper is structured as follows: section 2 deals with the basic concepts and properties of the concerned probability distributions, section 3 is on methodology and procedures towards the transformation of the triangular distribution and discussion of results; while in section 4, a concluding remark is made.

2 The Basic Concepts of the Probability Distributions

In this section, we present both the properties of the triangular and the continuous uniform distributions to be used in the later part of the work.

2.1 Some Moments and Properties of the Triangular Distribution (TD)

Suppose X is a random variable with parameters a, b and c such that

$$a : a \in (-\infty, \infty)$$
$$b : a \le b \le c$$
$$c : a < c$$

and a support , $a \le x \le b$

then, X is said to follow a triangular distribution $X \sim T(a,b,c)$ with the following properties:

Probability density function (pdf) $f(x)$ of X given as:

$$f(x)=\begin{cases} 0 \quad , & for \quad x<a \\ \dfrac{2(x-a)}{(c-a)(b-a)}, & for \quad a \le x \le b \\ \dfrac{2(c-x)}{(c-a)(c-b)}, & for \quad a \le x \le c \\ 0 \quad , & for \quad x>c \end{cases} \tag{1}$$

Cumulative Density Function (CDF) X :

$$F(x)=\begin{cases} 0, & for \quad x<a \\ \dfrac{(x-a)^2}{(c-a)(b-a)}, & for \quad a \le x \le b \\ 1-\dfrac{(c-x)^2}{(c-a)(c-b)}, & for \quad b < x \le c \\ 1, & for \quad c<x \end{cases} \tag{2}$$

Mean:

$$E(X)=\frac{a+b+c}{3} \tag{3}$$

Median:

$$X_{Median}=\begin{cases} a+\dfrac{\sqrt{(c-a)(b-a)}}{\sqrt{2}} & for \quad b \ge \dfrac{a+c}{2} \\ c-\dfrac{\sqrt{(c-a)(c-b)}}{\sqrt{2}} & for \quad b \le \dfrac{a+c}{2} \end{cases} \tag{4}$$

Most likely value

$$X_{Mode}=b \tag{5}$$

Variance:

$$Var(X) = \frac{a^2 + b^2 + c^2 - ac - ab - bc}{18} \tag{6}$$

Skewness:

$$X_{Skewness} = \frac{\sqrt{2}(a+c-2b)(2a-c-b)(a-2c+b)}{5(a^2 + c^2 + b^2 - ac - ab - bc)^{\frac{3}{2}}} \tag{7}$$

Kurtosis:

$$X_{Kurtosis} = -3/5 \tag{8}$$

2.2 Some Moments and Properties of the Continuous Uniform Distribution

A random variable Y over the interval $I = [a, b]$ is a continuous uniform distribution if it is *equally likely* to assume any value in I . Let $f(y)$ and $F(y)$ be the pdf and CDF of Y respectively, then:

$$f(y) = \begin{cases} \dfrac{1}{(\mathrm{b}-a)} & , \ a \le y \le b \\ 0 & , \ \text{otherwise} \end{cases} \tag{9}$$

$$F(y) = \begin{cases} 0, & \mathit{for} \quad \mathrm{y} < a \\ \dfrac{(y-a)}{(b-a)}, & \mathit{for} \ \mathrm{a} < y \le b \\ 1, & \mathit{for} \quad c \ge b \end{cases} \tag{10}$$

Remark 2.1: The following can easily be computed.

Mean:

$$Y_{mean} = \frac{b+a}{2} \tag{11}$$

Mode (can be any value in I) so we choose:

$$Y_{\mathrm{mod}e} = \frac{b+a}{2} \tag{12}$$

Median

$$Y_{median} = \frac{b+a}{2} \tag{13}$$

Variance

$$Var(Y) = \frac{(b-a)^2}{12} \tag{14}$$

Coefficient of skewness

$$Y_{c-skewness} = 0 \tag{15}$$

3 Methodology and the Modification of the Triangular Distribution

3.1 The Transformation of the Triangular Distribution (TTD)

We replace the most likely value b, with the average of the minimum and maximum of the Triangular distribution in order to modify the Triangular distribution, thus;

$$b = \frac{a+c}{2} \tag{16}$$

The resulting distribution will henceforth be referred to as the Transformed Triangular Distribution (TTD).

3.2 The Resulting Distribution TTD and Its Properties

Suppose X^* is the random variable associated with the TTD, $f^*(x)$ and $F^*(x)$ as the corresponding pdf and CDF respectively, then by using (9), one can easily obtain the following:

$$f^*(x) = \begin{cases} \dfrac{4(x-a)}{(c-a)^2} & , a \le x \le \dfrac{a+c}{2} \\ \dfrac{4(\mathrm{c}-\mathrm{x})}{(c-a)^2} & , \dfrac{a+c}{2} \le x \le c \\ 0 & , otherwise \end{cases} \tag{17}$$

and it is easy to show that:

$$F^*(x)=\begin{cases}0, & for \quad x<a\\ \dfrac{2(x-a)^2}{(c-a)^2}, & for \quad a\le x\le \dfrac{a+c}{2}\\ 1-\dfrac{2(c-x)^2}{(c-a)^2}, & for \quad \dfrac{a+c}{2}<x\le c\\ 1, & for \quad c\le x\end{cases} \tag{18}$$

Remark 3.1 validation of the probability density function of $f^*(x)$

To validate the pdf of the TTD, we need to show that:

$$\int_a^{\frac{a+c}{2}} f^*(x)dx=1 \tag{19}$$

Proof: By definition,

$$\int_a^c f^*(x)dx=\int_a^{\frac{a+c}{2}}\frac{4(x-a)}{(c-a)^2}dx+\int_{\frac{a+c}{2}}^{c}\frac{4(c-x)}{(c-a)^2}dx$$

$$=\frac{4}{(c-a)^2}\left\{\int_a^{\frac{a+c}{2}}(x-a)dx+\int_{\frac{a+c}{2}}^{c}(c-a)dx\right\}$$

$$=\frac{4}{(c-a)^2}\{A+B\} \tag{20}$$

where

$$A=\left(\frac{x^2}{2}-ax\right)\Bigg|_a^{\frac{a+c}{2}}=\frac{(a-c)^2}{8} \quad \& \quad B=\left(cx-\frac{x^2}{2}\right)\Bigg|_{\frac{a+c}{2}}^{c}=\frac{(c-a)^2}{8} \tag{21}$$

Thus,

$$\int_a^c f^*(x)dx=\frac{4}{(c-a)^2}\{A+B\} \tag{22}$$
$$=1$$

Since

$$A+B=\frac{(a-c)^2}{4} \quad \& \quad (a-c)^2\equiv(c-a)^2,\ a\ne c \tag{23}$$

Showing that $f^*(x)$ is indeed a valid pdf. $\boxdot$

For the rest of the moments and properties we shall often refer to (17), as such (16) in (3) gives:

Mean:

$$X^*_{mean} = \frac{a + \frac{a+c}{2} + c}{3} = \frac{a+c}{2} \geq \frac{b+a}{2} \tag{24}$$

Thus,

$$X^*_{mean} \geq Y_{mean}$$

showing that:

$$Y_{mean} \square X^*_{mean} \tag{25}$$

The Median of TTD using the extreme values:

Substitute equation (16) in (4) gives:

$$X^*_{median} = a + \sqrt{\frac{(c-a)(\frac{a+c}{2} - a)}{2}} = a + \sqrt{\frac{(c-a)(c-a)}{4}} = \frac{a+c}{2} \geq Y_{median} \tag{26}$$

showing that:

$$Y_{median} \square X^*_{median} \tag{27}$$

Similarly

$$Median = c - \sqrt{\frac{(c-a)(c-(\frac{a+c}{2})}{2}} = c - \sqrt{\frac{(c-a)(c-a)}{4}} = \frac{a+c}{2}$$

showing the same result in (27).

The Mode of TTD (can be any value in I):
In this case of mode, choose b such that:

$$X^*_{\text{mod}e} = b = \frac{a+c}{2} \tag{28}$$
$$\geq Y_{\text{mod}e}$$

showing that:

$$Y_{\text{mod}e} \square X^*_{\text{mod}e} \tag{29}$$

The variance of the TTD:
Substitute (16) in (6) gives:

$$Var(X^*) = \frac{a^2 + (\frac{a+c}{2})^2 + c^2 - ac - a(\frac{a+c}{2}) - c(\frac{a+c}{2})}{18}$$
$$= \frac{(c-a)^2}{24} \tag{30}$$
$$\geq Var(Y)$$

showing that:

$$Var(Y) \square Var(X^*) \tag{31}$$

The skewness of the TTD:
From (16)

$$b = \frac{a+c}{2} = 2b = a+c \tag{32}$$

Therefore, substituting (16) and (32) in (7) gives:

$$X^*_{Skewness} = \frac{\sqrt{2}\left(2b-2b\right)\left(2a-c-\left(\frac{a+c}{2}\right)\right)\left(a-2c+\left(\frac{a+c}{2}\right)\right)}{5\left(a^2+c^2+\frac{\left(a+c\right)^2}{4}-ac-a\frac{\left(a+c\right)}{2}-c\frac{\left(a+c\right)}{2}\right)^{\frac{3}{2}}} \tag{33}$$
$$= 0 = Y_{c-skewness}$$

4 Conclusion

In this paper, we have showed that when the mode of a triangular distribution is the average of the extreme values, then the resulting distribution referred as transformed triangular distribution (TTD) is a probability function that can be reasonably approximated, simulated and modeled by the continuous uniform distribution. The limit theorem takes care of the behavior of the distribution at large sample.

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Competing Interests

Authors have declared that no competing interests exist.

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