On a Modified Ratio of Exponential Distributions

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Abstract: In this paper, the ratio of two independent exponential random variables is studied and another two-parameter probability model representing the modified ratio of exponential distributions (MRED) is defined. This new model is proposed in modeling the survival of patients undergoing surgery.

Keywords: Exponential, Modeling, Modified, Survival

1.0 Introduction:

The distributions of the product and ratio of independent random variables are of great importance in many areas of sciences. An important example of ratios of random variables is the stress-strength model in the context of reliability. For a given two independent random variables, X and Y, the distribution of the ratio $\frac{X}{Y}$ as explained by Nadarajah and Kotz (2007) could represent (a) the relative strength of two different signals in communication theory, (b) the relative safety of navigation in ocean engineering, (c) the relative popularity of two different commodities in finance, also see Annavajjala et al (2010), Hamedani (2013), Nadarajah,(2005a, 2005b, 2006), Mdziniso (2012) and Pham-Gia (2000).

According to Steven (2012), for two independent exponentially distributed random variables X and Y, both with rate parameter λ, the distribution of the ratio $U = \frac{X}{Y}$ is given by;

$$f(u) = \frac{1}{(u+1)^2} \qquad (1)$$

where $u \in (0, \infty)$ because $x \in (0, \infty)$ and $y \in (0, \infty)$. He confirmed that the expected value of the distribution in Equation (1) is undefined [4].

This article seeks to extend the work done by Steven (2012) by verifying the validity of his model, showing the shape of the model, providing an explicit expression for the cumulative density function (c.d.f), the survival function and hazard function.

Conversely, the model provided by Steven (2012) does not contain any model parameter. This is because the two random variables considered both have same rate parameter λ . Hence, it is of interest in this article to derive a two-parameter model for the ratio of two independent exponential random variables with different rate parameters say λ_1 and λ_2 respectively and to interpret the parameters.

2.0 Methodology

2.1 Validity of Steven's (2012) model

This suffices that $\int_0^\infty f(u)du = 1$

$$f(u) = \frac{1}{(u+1)^2}$$

$$\int_0^\infty f(u)du = \int_0^\infty \frac{1}{(u+1)^2} du$$

Let $= u + 1$, if $u = 0,$ then, $t = 1$, $\frac{dt}{du} = 1$, $du = dt$

$$\int_0^\infty f(u)du = \int_1^\infty \frac{1}{t^2} dt$$

$$= 1$$

Hence, the model provided by Steven (2012) is a valid p.d.f

2.2 The Shape of Steven's (2012) model

The shape of the model in Equation (1) is represented graphically as below;

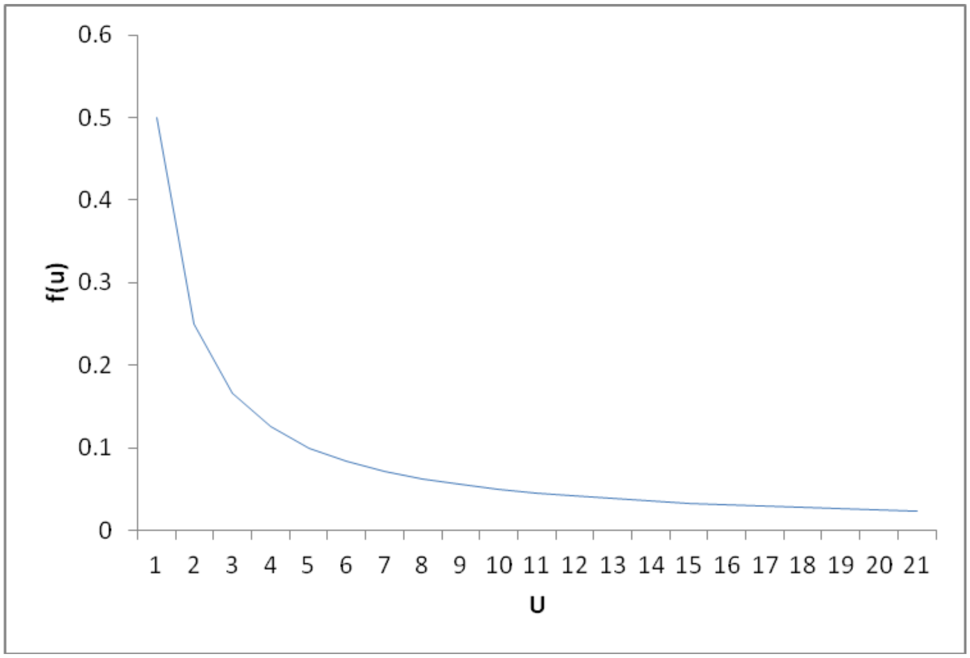

Fig. 1: Graph of the pdf of U, $u \in (0, 20)$

It can be seen from the graph that the model is positively skewed.

2.3 Cumulative Density Function (CDF)

The corresponding cdf is expressed as

$$F(u) = P(U \leq u) = \int_0^u f(t)dt$$

$$= \int_0^u \frac{1}{(t+1)^2} dt$$

$$= \left[-\frac{1}{t+1}\right]_0^u$$

$$F(u) = \frac{u}{u+1} \qquad (2)$$

It can be seen from Equation (2) that $\lim_{u \to +\infty} F(u) \cong 1$

The cumulative density function is represented graphically as below;

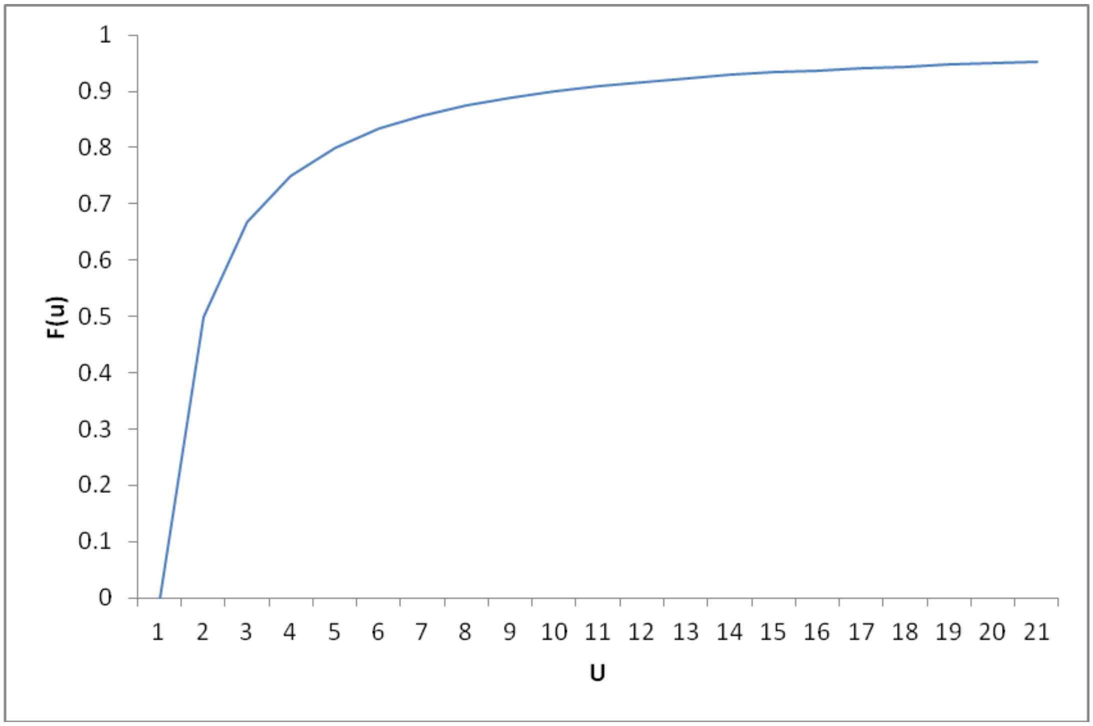

Fig. 2: Graph of the CDF of U, $u \in (0, 20)$

2.4 Survival Function:

By definition, the survival function for a random variable Z is defined as;

$$S(u) = P(U > u)$$

$$S(u) = 1 - P(U \leq u) = 1 - F(u)$$

$$S(u) = \frac{1}{u+1} \tag{3}$$

As $u \to 0$; $\lim_{u \to 0} S(u) = 1$

As $u \to \infty$; $\lim_{u \to \infty} S(u) = 0$

2.5 Hazard Function:

Mathematically, the hazard function for a random variable U is given by;

$$h(u) = \frac{f(u)}{1 - F(u)}$$

$$h(u) = \frac{1}{u+1} \tag{4}$$

As $u \to 0$; $\lim_{u \to 0} h(u) = 1$

As $u \to \infty$; $\lim_{u \to \infty} h(u) = 0$

These results above justify that $h(u) \geq 0$

The Survival function and Hazard function is represented graphically as below;

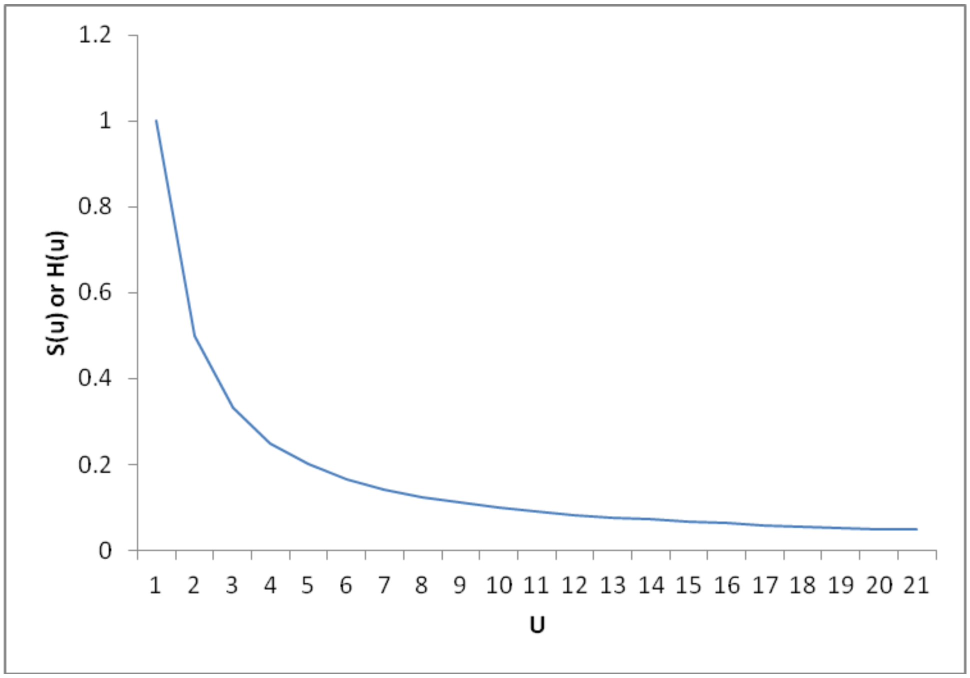

Fig. 3: The Survival and Hazard Function of U, $u \in (0, 20)$

It can be seen that the expressions for the survival function and hazard function for Steven's (2012) model as derived in Equations (3) and (4) are the same. It can also be observed that the graph for the survival function and hazard function for Steven's (2012) are perfectly the same. The implication is that the distribution would be appropriate to model the survival of a patient undergoing surgery.

2.6 The Modified Ratio of Exponential Distributions (MRED)

Let U and V denote two independent exponential random variables with parameters λ_1 and λ_2 respectively. That is, $U \sim Exp(\lambda_1)$ and $V \sim Exp(\lambda_2)$. Then,

$$f(u) = \lambda_1 e^{-\lambda_1 u} ; \quad \lambda_1 > 0$$

$$f(v) = \lambda_2 e^{-\lambda_2 v} ; \quad \lambda_2 > 0$$

Let $X = \frac{U}{V}$ denote another random variable representing the ratio of the random variables U and V. Using the method of transformation, the pdf of X is derived as;

$X = \frac{U}{V}$, let $Y = V$, then, $V = Y$ and $U = XY$

$\frac{du}{dx} = y$, $\frac{du}{dy} = x$, $\frac{dv}{dx} = \mathbf{0}$, and $\frac{dv}{dy} = \mathbf{1}$

The Jacobian matrix of transformation is given by; $J = \begin{vmatrix} \frac{du}{dx} & \frac{du}{dy} \\ \frac{dv}{dx} & \frac{dv}{dy} \end{vmatrix}$

$J = \begin{vmatrix} y & x \\ \mathbf{0} & \mathbf{1} \end{vmatrix} = y$

The joint pdf; $f(u, v) = f(u)f(v)$ is given by;

$f(u, v) = \lambda_1 \lambda_2 e^{-(\lambda_1 u + \lambda_2 v)}$

For $U = XY$ and $V = Y$,

$f(xy, y) = \lambda_1 \lambda_2 e^{-(\lambda_1 x + \lambda_2)y}$

The marginal pdf of random variable X is therefore given by;

$f(x) = \int_{-\infty}^{\infty} f(xy, y). |J| dy = \int_{0}^{\infty} f(xy, y). |J| dy$

$= \lambda_1 \lambda_2 \int_{0}^{\infty} e^{-(\lambda_1 x + \lambda_2)y} . y \, dy$

Let $\theta = (\lambda_1 x + \lambda_2)$

$f(x) = \lambda_1 \lambda_2 \int_{0}^{\infty} y \, e^{-\theta y} \, dy$

By definition, $E(X) = \int_{0}^{\infty} \theta x \, e^{-\theta x} \, dx = \frac{\mathbf{1}}{\theta}$ (the mean of an exponential distribution with parameter θ)

$f(x) = \frac{\mathbf{1}}{\theta} . \lambda_1 \lambda_2 \frac{.1}{\theta}$

$$= \frac{\lambda_1 \lambda_2}{\theta^2}$$

$$f(x) = \frac{\lambda_1 \lambda_2}{(\lambda_1 x + \lambda_2)^2} \quad (5)$$

where $\lambda_1 > 0, \lambda_2 > 0 \; and \; x \geq 0$.

Equation (5) above is the pdf of the Modified Ratio of Exponential Distributions (MRED). It can otherwise be represented by $X \sim MRED(\lambda_1, \lambda_2)$.

The shape of the MRED with different parameters of λ_1 and λ_2 is shown graphically as below;

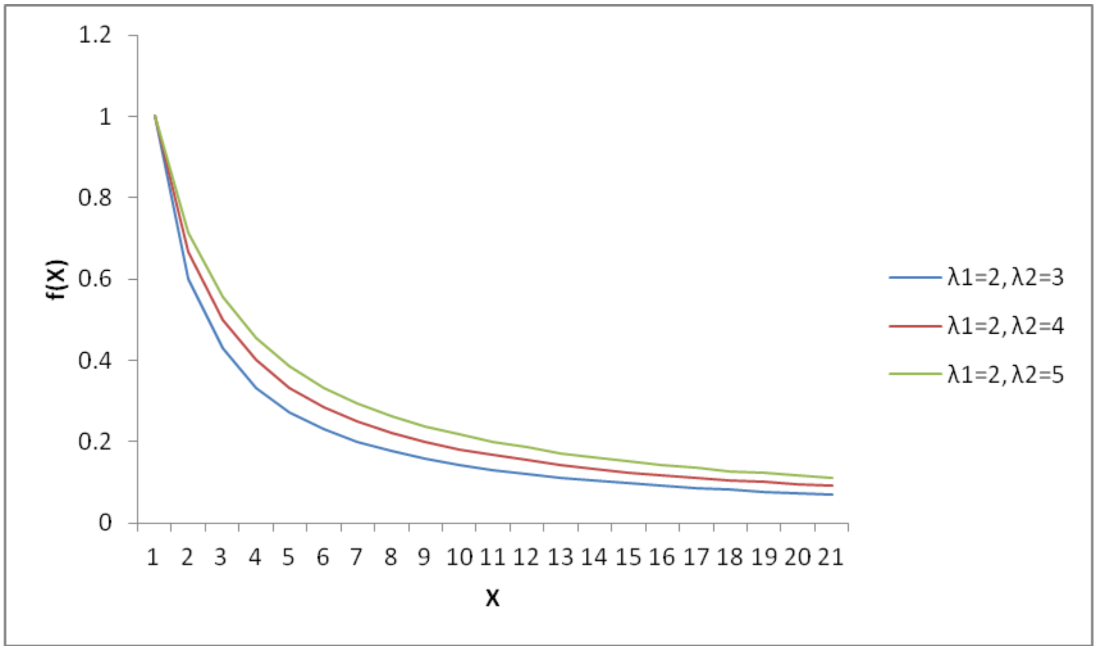

Fig. 4: Graph of the Modified Ratio of Exponential Distributions

The model is positively skewed and the two parameters; λ_1 and λ_2 are both location parameters.

It was observed that $\int_0^\infty f(x)dx = 1$. That is,

$$\int_0^\infty \frac{\lambda_1 \lambda_2}{(\lambda_1 x + \lambda_2)^2} dx = 1$$

Hence, the Modified Ratio of Exponential Distributions is a valid pdf.

The corresponding cumulative density function is given by;

$$F(x) = \int_0^x f(t)dt$$

$$= \int_0^x \frac{\lambda_1 \lambda_2}{(\lambda_1 t + \lambda_2)^2} dt$$

$$F(x) = \frac{\lambda_1 x}{\lambda_1 x + \lambda_2} \quad (6)$$

3.0 Conclusion:

In this paper, the distribution of the ratio of two (independent) exponential random variables that was defined by Steven (2012) is being explored further and it was found that the survival function and the hazard function for the model are the same. Hence, the model would be useful in biological or medical sciences to model the survival of patients undergoing surgery in any disease. Another two-parameter probability model (Modified Ratio of Exponential Distributions) is being defined in this paper and the parameters are found to be location parameters.

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